Lab 1: Damped, Driven Harmonic Oscillator - PHY300 Spring - 2014

I. INTRODUCTION

The purpose of this experiment is to study the resonant properties of a driven, damped harmonic oscillator. This type of motion is characteristic of many physical phenomena. For example, radiation from atoms in a laser can be described with the same equations developed here...only the names of the variables are changed. We use a sheet of flat steel that experiences a force trying to straighten it when it is bent. Since this force is proportional to the amount of bending, it acts like a linear spring, so we will analyze the problem this way.

In this experiment a mirror is mounted on the end of a flat piece of springy steel and set into vibration as shown on the left side of Fig. 1. A beam of laser light is bounced off the mirror onto the wall, and when the mirror moves, so does the light spot on the wall. When the mirror vibrates, the light spot does also, and its motion provides information about the position of the mirror. It appears as a continuous line if the vibration frequency is high enough.

CAUTION: Laser beams are potentially dangerous and can cause blindness if looked at directly. Do not look directly into the laser beam.

Question 1: Show that a beam of light incident on the mirror as shown on the left side of Fig. 1 will be deviated by reflection through an angle twice as large as the change of the mirror angle. That is, for the case shown, show that the reflected and incoming beam make an angle 2θ.

Question 2: How does the position of the spot on the wall depend on the position of the mirror? Clearly the angle changes as the spring vibrates, so the spot moves. But also the position of the mirror changes because it not only turns, but also moves. Make a careful analysis and determine the important parameters.

II. THEORY

The right side of Fig. 1 shows the idealization of the linear oscillator we will use to analyze this experiment. Consider a mass M located at a distance x_1 from a fixed barrier connected to it with a stretched spring and held by another spring whose end can be driven at any frequency ω with peak-to-peak amplitude $2s$. When $s = 0$, the position of the right end of the right-hand spring is $x_1 + x_2$ as shown in Fig. 1. The sum of the forces on the mass from the springs is $-k(x_1 - p + x) + k(x_2 - p - x)$, where k is the spring constant for both springs (assumed equal), p is their unstretched length, and x is the distance of the mass from its equilibrium point x_1. If x_1 and x_2 are equal, which is expected if the spring constants and unstretched lengths are the same, then the sum of the forces on the mass from the springs is simply $-2kx$. If the mass is moving, it experiences friction, and for simplicity we'll consider the case where the friction force is proportional to dx/dt, and so the sum of the forces on it is $-2kx - b(dx/dt)$ which must equal the mass times the acceleration, $M(d^2x/dt^2)$. The velocity-dependent friction force is $-b(dx/dt) = -bv$, where v is the velocity.

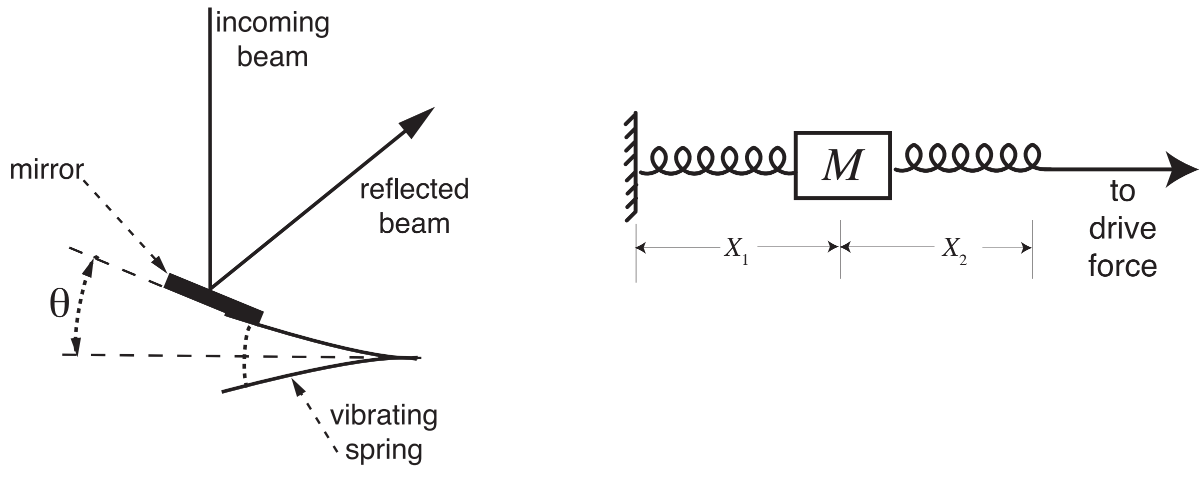

FIG. 1: The left side is a schematic diagram of the vibrating sheet metal spring and the light beam reflecting from the mirror mounted on it. The right side shows the idealization of this oscillator as a mass-spring oscillator. The mass is at equilibrium at position x_1 when it is at rest. Oscillations occur about x_1 at the driving frequency ω or, in the case of zero drive, at the resonant frequency ω_0.

Under these conditions, the motion of the mass when displaced from equilibrium by A is simply that of a damped oscillator, $x = A\cos\omega_0 t\, e^{-\gamma t/2}$ where $\omega_0 = \sqrt{K/M}, K = 2k$, and $\gamma = b/M$. Later we will discuss your measurement of this phenomenon. Now suppose that the right hand end of the right hand spring is vibrated so instead of the end being fixed at $x_1 + x_2$ its position is given by $x_1 + x_2 + s\sin\omega t$. Then the sum of the forces includes the driving force, and the equation of motion becomes

$$M\frac{d^2x}{dt^2} = -Kx - b\frac{dx}{dt} + F_0\cos\omega t \tag{1}$$

where $F_0 = Ks$. Equation 1 is the very famous damped, forced oscillator equation that reappears over and over in the physical sciences.

There are many possible solutions to this equation, but only those that correspond to physical reality are sought. Experimentally it is clear that the mass will oscillate at the driving frequency that can be varied over a wide range. The motion of the mass differs in phase from the drive even though its frequency is the same, but this will be hard to see in this experiment. A difference in phase means that the mass does not always move in the direction of the applied force, but may sometimes move in the opposite direction.

Try a solution of the equation in which the force is out of phase with the motion by ϕ. For simplicity change the phase of the force instead of the phase of the motion (this clearly is the same as changing the phase of the motion by the negative of the force phase change) so $x = A\cos\omega t$ and then Eq. 1 will become

$$-MA\omega^2\cos\omega t = -AK\cos\omega t + bA\omega\sin\omega t + F_0\cos(\omega t + \phi) \tag{2}$$

where ϕ is the phase difference. This equation must hold for **all** values of time, so we can choose any time to evaluate it. At $t = 0$, and at $t = \pi/2\omega$, we find

$$MA\omega^2 = +AK - F_0\cos\phi \qquad \text{and} \qquad 0 = bA\omega + F_0\cos(\pi/2 + \phi) = bA\omega - F_0\sin\phi \tag{3}$$

since $\cos(\pi/2 + \phi) = -\sin\phi$. Solve the second part of Eq. 3 for $\sin\phi$ and calculate $\cos\phi$ using $\cos^2\phi = 1 - \sin^2\phi$ and substitute it into the first Eq. 3. Isolate the radical on one side of the equation and then square both sides. The result is $A^2(K - M\omega^2)^2 = F_0^2 - (bA\omega)^2$ from which we obtain

$$A = \frac{F_0/M}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}} \tag{4}$$

where the frequency $\omega_0 = \sqrt{K/M}$ is the oscillation frequency when there is zero driving force. This is called the natural frequency of the oscillator, or the resonance frequency.

Question 3: Show that A in Eq. 4 has the dimension of distance. What is the required dimension of $\gamma = b/M$ for this to be true? Show that this is indeed the dimension of γ.

Question 4: In this derivation, sines and cosines were used in place of exponential notation, and the consequence was considerable extra algebra. Replace the $\cos(\omega t + \phi)$ term by $\Re(e^{i(\omega t+\phi)})$, replace the trial expression for x with $\Re(Ae^{i\omega t})$, separate the real and imaginary parts of the result, and solve for A. (Here $\Re$ and $\Im$ refer to real and imaginary parts respectively.) Compare your expression for A with Eq. 4.

Although Eq. 4 may look a little complicated, it actually has a very beautiful and simple form. The denominator clearly has a minimum when the driving frequency is equal to the resonance frequency, and this makes A have a maximum. If the energy damping rate $\gamma = b/M$ is very small (note that this is twice the amplitude damping rate where $x = A\cos\omega_0 t\, e^{-\gamma t/2}$), the maximum value of A becomes very large. The graph of A versus driving frequency is approximately symmetrical about the resonance frequency for small γ because of the square of the difference of the squares that appears under the radical.

You can also take Eq's. 3 above, solve them for the sine and cosine terms, and divide, thereby eliminating F_0. Substituting ω_0^2 for K/M, one obtains

$$\tan\phi = \frac{b\omega}{M(\omega_0^2 - \omega^2)}. \tag{5}$$

Question 5: Continuing your work on question 4, find ϕ using exponential notation.

Equations 4 and 5 describe the motion you are going to study in the laboratory. Notice that as the driving frequency approaches the resonance frequency from either above or below it, the amplitude of the oscillation A becomes very large. For $\omega = \omega_0$, $A = F_0/b\omega_0$ and diverges as $b \rightarrow 0$. If there were no friction the amplitude would become infinitely large, but before this happened the mirror would hit the driving magnet (see below) or spring would break. The amplitude of oscillation at a given damping is always near a maximum when the system is driven at its resonance frequency.

Also notice that if the driving frequency is nearly ω_0, $\tan\phi$ is very large and thus ϕ is approximately $\pm\pi/2$. For example, substitute $\phi = -\pi/2$ into Eq. 2 and find that the driving force is $-F_0 \sin\omega t \times \sin(-\pi/2)$ which is the same as $F_0 \sin\omega t$. The oscillation motion is almost exactly $\pi/2$ out of phase with the driving force!!! Its $\pi/2$ phase changes sign when ω passes through ω_0. Also notice that if the driving frequency is very much larger than the resonant frequency, then $\tan\phi$ becomes a very small negative number and this means $\phi \sim -\pi$. Then the motion is almost exactly out of phase with the drive; when the spring pulls the mass moves away and when the spring pushes the mass moves against the push. This phenomenon is not readily observable with your oscillator, and so you need to do a separate, less precise experiment to observe it, as discussed below.

III. PROCEDURE

In the first part of this experiment you will measure the damping time $1/\gamma$ by watching the decay of the oscillator using the laser spot on the wall. First, align the laser beam so it is pointing directly away from the wall and is incident nearly perpendicular to the mirror, but with enough of an angle so that the reflected beam misses the laser and hits the wall behind it. Set the spot on the wall at a convenient place for measurement. Then carefully displace the mirror by a few mm by bending the spring and releasing it. The spot on the wall should smear out into a line several cm long, and then slowly shrink back to a spot.

To determine γ you need to measure the time dependence of the length of this line, and since the decay time is only a few seconds, this is a bit difficult. Practice with your lab partner measuring this length after each second, and plot the results. Repeat the experiment several times. Plot your results on semi-log paper and use the slope to find γ, or use a least squares fit to $e^{-\gamma t}$. Be sure to propagate the uncertainties.

Question 6: Show that for an exponential decay, the measured decay constant is **independent** of the initial displacement of the mirror.

In the second part of this experiment, you will measure a resonance curve that should look like that of Fig. 2. This is just practice in getting used to the apparatus, because the work needed to answer question 7, a *required* part of this lab, will take a longer time. There is no known answer to that question, and so it's a matter of pure investigation for you to figure out. In this sense, it truly mimics real laboratory research.

The steel spring is magnetic, and to drive it we apply an oscillating magnetic field using an electronic oscillator whose frequency and amplitude can be varied. The circuit is exquisitely simple - just connect the magnet's leads to the oscillator's plug with the alligator clip leads. Turn on the oscillator, set its frequency somewhere around 10 - 20 Hz, and adjust the amplitude so the laser spot on the wall is smeared into a line a few cm long. Then carefully maximize the length of this line by slowly varying the oscillator frequency. The line may stretch out very long, and you should reduce the drive amplitude until it is only a few cm long.

Now connect the output of the oscillator also to the frequency counter so you can determine the drive frequency simultaneously with observing the line on the wall. Then play with the frequency. Away from resonance the line will be short, only a few mm, and you'll have to measure this without changing the drive, so try to determine the best value for the amplitude.

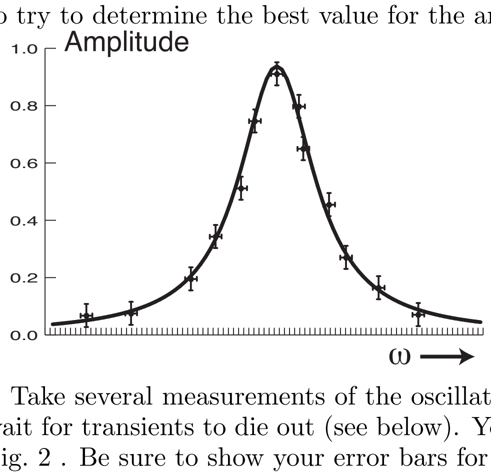

FIG. 2. The "measured" amplitudes are plotted against the driving frequency, and each data point has error bars in both dimensions. Each amplitude was measured several times, the results were averaged, and the uncertainty is the standard deviation from the average. The plotted values have all been divided by the measured maximum value. The (smaller) error bars on the frequency derive from estimates of the precision of reading the counter. The smooth curve was fitted to the data.

Take several measurements of the oscillation amplitude at each of several driving frequencies. Be sure to wait for transients to die out (see below). You should be able to plot resonance curves like the one shown in Fig. 2 . Be sure to show your error bars for both amplitude and frequency.

You can also determine γ from this measurement, and you should do so. Solve Eq. 4 for the case $A = A_{max}/2$ where $A_{max} = F_0/\omega_0 b$ is the amplitude on resonance, $\omega = \omega_0$. There are two solutions, corresponding to the frequencies where the measured curve is at half of its maximum, and their difference is the full width at half maximum of the curve (FWHM is the standard abbreviation). Do the algebra to show that the FWHM = γ, determine this value from your measurements, and compare the result with that of the first part of the experiment. Be sure to propagate the uncertainties, and interpret your comparison of the two methods in the light of these values.

Now change something. The easiest thing to do is add a mass to the vibrating spring by simply putting on an alligator clip. Repeat the experiment, and observe what happens to the resonance frequency.

Question 7: Does the position of the alligator clip matter? Using the same alligator clip, take several resonance curves and plot the resonance frequency vs. position of the clip. You'll have to figure out some accurate way to measure this position. This problem has no simple solution - we don't know the answer. Only you will find it out. There is no "**right**" answer for this plot. Be sure to show your error bars.

For the third part of this experiment you need to observe the relation between the phase of oscillation and the driving frequency. To do this, simply hang a weight on a string about 1 m long to make a simple pendulum, and swing it with your hand. If you move VERY slowly, the weight will simply follow under you hand, in phase. If you oscillate your hand very fast, the weight will move in exactly the opposite direction, with an amplitude smaller than the motion of your hand. When you drive it at resonance, it will be clear that the motion is neither in phase nor out of phase, and careful observation will show that the phase is about halfway between these limits.

IV. HINTS AND KINKS DEPARTMENT

The primary problem that you will face in this experiment arises from the transients of motion in the oscillation. The solutions to the equation derived earlier are valid only after a long period of time has elapsed from starting. They do not describe the motion at the beginning, nor do they account for any possible motion at any frequency other than the driving frequency. In class you have been told that if the oscillation has a component of motion at any frequency other than the driving frequency, then the total motion will be the sum of the motion at the driving frequency and the oscillation at the other frequency (call it ω'). This sum can be written as $A\cos\omega t + B\cos\omega' t$ which becomes, with the use of a trig identity,

$$2A\cos\left[(\omega+\omega')t/2\right]\cos\left[(\omega-\omega')t/2\right] + (B-A)\cos\omega' t \tag{6}$$

A plot of this motion shows one oscillation at the average of the frequencies contained in an envelope at the difference of the frequencies. For a while the oscillation will appear at close to the driving frequency but then the amplitude of the oscillation will decrease significantly. The laser spot will then wiggle a little bit and begin oscillating again. This peculiar behavior will persist until the oscillation at ω' has died away (i.e., $B = 0$ above). The loss of energy in the system at ω' depends on ω' and may take several seconds before becoming complete. If you try to measure the amplitude of an oscillation after a frequency change without waiting for the energy at the old frequency to be lost through friction, the results will be that the amplitude is not constant but varies with time. One way to tell if there is no energy left at any frequency other than the driving frequency is to observe the amplitude for 20 or 30 seconds. If it remains constant, you can take that measurement as the amplitude at that frequency.

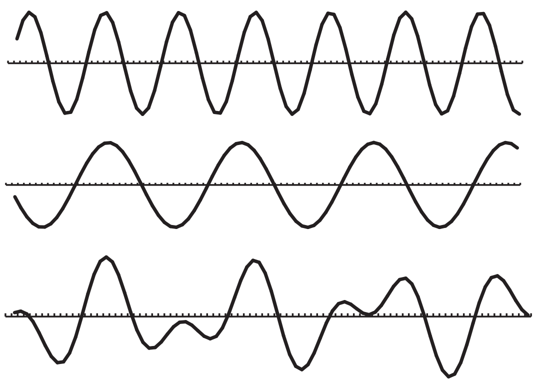

FIG. 3: The upper two curves show oscillations at quite different frequencies with approximately equal amplitudes. Their sum is plotted in the bottom curve on a different scale. It is clear that the superposition motion of the sum shows no simple characteristics, and that its amplitude varies with time.