Propagation of Errors

In order to obtain the desired final result in an experiment it is usually necessary to add, subtract, multiply, and/or divide various measured quantities. It will be assumed that a probable fractional uncertainty has been assigned to each measured quantity (found by dividing the uncertainty by the most probable value of the quantity). If sufficient data have been taken to calculate an arithmetic mean (average) and find a distribution width, this fractional error will be the width divided by the arithmetic mean. If only one measurement of the quantity has been made, this fractional error will be the student's estimate of the probable uncertainty divided by this measurement. If sufficient effort has not been made to eliminate systematic errors, a probable uncertainty should be assigned to take them into account as well as random errors. It then becomes necessary to determine the way the probable uncertainties in the several different measured quantities combine to determine the probable uncertainty in the final result.

1. In addition, one adds the numerical values of the probable uncertainties in the quantities in order to determine the probable uncertainty in the sum. For example, suppose you need to add the 3 numbers N_1, N_2, N_3. Each of these has a uncertainty $\pm e_1$, $\pm e_2$, and $\pm e_3$, presumably much smaller than their relevant N-value, then you need the sum: $(N_1 \pm e_1) + (N_2 \pm e_2) + (N_3 \pm e_3)$. The maximum value obtainable would be $N_1+N_2+N_3+(e_1+e_2+e_3)$ and the minimum would be $N_1+N_2+N_3-(e_1+e_2+e_3)$. It is possible that + errors will cancel the - errors, giving no error in the sum. Since one cannot determine whether or not this is the case the rule for now is to be conservative and state the sum as $N_1 + N_2 + N_3 \pm (e_1 + e_2 + e_3)$.

2. In subtraction, one does likewise: $(N_1 \pm e_1) - (N_2 \pm e_2) = N_1 - N_2 \pm (e_1 + e_2)$

3. In multiplication, follow the same algebraic procedure as you would for multiplication of polynomials (remember $B \ll A$ and $D \ll C$): $(A \pm B) \times (C \pm D) = AC \pm BC \pm AD \pm BD$. Since BD is much smaller than either BC or AD, it can be neglected. If you do that, you can re-write the answer as: $AC \pm AC(B/A + D/C) = AC\,[1 \pm (B/A + D/C)]$ and note that B/A is the fractional error in A and D/C is the fractional error in C. It is clear then that you can simply add the fractional errors of each of the quantities and get the fractional error of the product. This technique of adding fractional errors can be extended to products of more than 2 numbers and also to quotients as shown below.

4. For the case of division: $(A \pm B)/(C \pm D) = A/C \pm (A/C)(B/A + D/C) = A/C \pm B/C \pm AD/C^2$. You can prove this by saying that the quotient must have the form $X \pm Y$ and cross-multiplying. The resulting equation is $A \pm B = (C \pm D)(X \pm Y) = CX \pm CY \pm DX \pm DY$. Now it's clear that $X = A/C$ because if the measurements were perfect, both B and D would be zero. Then solve for Y assuming D is small compared to C and can be dropped in the sum $C + D$.

In the examples below, let $A = 25 \pm 2$, $B = 5 \pm 1$, $C = 40 \pm 3$, $D = 20 \pm 1$, and $E = 30 \pm 5$ and evaluate:

1. $A + B$	2. AB
3. A/B	4.ABC
5. AB/CDE	6. $A + B - C + D - E$
7. $AB + C^2/A$	8. $\sqrt{AB}$

In order to do #8, just assume the answer has the form $X \pm Y$ and then square both sides. Solve for Y assuming $X = \sqrt{125}$.